2.3 Continuity

What you will learn about . . .

- Continuity at a Point
- Continuous Functions
- Algebraic Combinations
- Composites
- Intermediate Value Theorem for Continuous Functions

and why . . .

Continuous functions are used to describe how a body moves through space and how the speed of a chemical reaction changes with time.

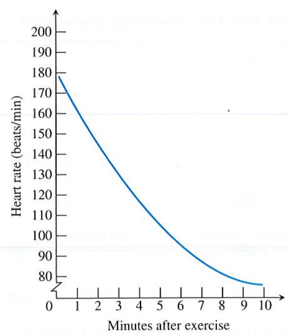


Figure 2.16 How the heartbeat returns to a normal rate after running.

Continuity at a Point

When we plot function values generated in the laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the times we did not measure (Figure 2.16). In doing so, we are assuming that we are working with a *continuous function,* a function whose outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between. Any function $y = f(x)$ whose graph can be sketched in one continuous motion without lifting the pencil is an example of a continuous function.

Continuous functions are those we use to find a planet's closest point of approach to the sun or the peak concentration of antibodies in blood plasma. They are also the functions we use to describe how a body moves through space or how the speed of a chemical reaction changes with time. In fact, so many physical processes proceed continuously that throughout the 18th and 19th centuries it rarely occurred to anyone to look for any other kind of behavior. It came as a surprise when the physicists of the 1920s discovered that light comes in particles and that heated atoms emit light at discrete frequencies (Figure 2.17). As a result of these and other discoveries, and because of the heavy use of discontinuous functions in computer science, statistics, and mathematical modeling, the issue of continuity has become one of practical as well as theoretical importance.

To understand continuity, we need to consider a function like the one in Figure 2.18, whose limits we investigated in Example 8, Section 2.1.

Figure 2.17 The laser was developed as a result of an understanding of the nature of the atom.

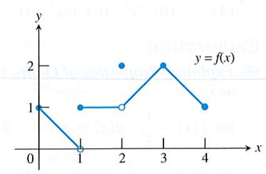


Figure 2.18 The function is continuous on $[0, 4]$ except at $x = 1$ and $x = 2$. (Example 1)

EXAMPLE 1 Investigating Continuity

Find the points at which the function f in Figure 2.18 is continuous, and the points at which f is discontinuous.

SOLUTION

The function f is continuous at every point in its domain $[0, 4]$ except at $x = 1$ and $x = 2$. At these points there are breaks in the graph. Note the relationship between the limit of f and the value of f at each point of the function's domain.

Points at which f is continuous:

At $x = 0$, $\quad \lim_{x \to 0^+} f(x) = f(0)$.

At $x = 4$, $\quad \lim_{x \to 4^-} f(x) = f(4)$.

At $0 < c < 4, c \neq 1, 2$, $\quad \lim_{x \to c} f(x) = f(c)$.

continued

Points at which f is discontinuous:

At $x = 1$, $\quad \lim_{x \to 1} f(x)$ does not exist.

At $x = 2$, $\quad \lim_{x \to 2} f(x) = 1$, but $1 \neq f(2)$.

At $c < 0, c > 4$, $\quad$ these points are not in the domain of f.

Now Try Exercise 5.

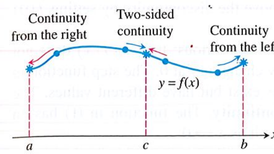


Figure 2.19 Continuity at points a, b, and c for a function $y = f(x)$ that is continuous on the interval $[a, b]$.

To define continuity at a point in a function's domain, we need to define continuity at an interior point (which involves a two-sided limit) and continuity at an endpoint (which involves a one-sided limit) (Figure 2.19).

DEFINITION Continuity at a Point

Interior Point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

If a function f is not continuous at a point c, we say that f is **discontinuous** at c and c is a **point of discontinuity** of f. Note that c need not be in the domain of f.

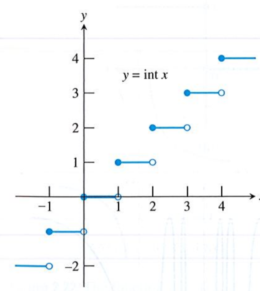


Figure 2.20 The function int x is continuous at every noninteger point. (Example 2)

EXAMPLE 2 Finding Points of Continuity and Discontinuity

Find the points of continuity and the points of discontinuity of the greatest integer function (Figure 2.20).

SOLUTION

For the function to be continuous at $x = c$, the limit as $x \to c$ must exist and must equal the value of the function at $x = c$. The greatest integer function is discontinuous at every integer. For example,

$$\lim_{x \to 3^-} \text{int } x = 2 \quad \text{and} \quad \lim_{x \to 3^+} \text{int } x = 3$$

so the limit as $x \to 3$ does not exist. Notice that int $3 = 3$. In general, if n is any integer,

$$\lim_{x \to n^-} \text{int } x = n - 1 \quad \text{and} \quad \lim_{x \to n^+} \text{int } x = n,$$

so the limit as $x \to n$ does not exist.

The greatest integer function is continuous at every other real number. For example,

$$\lim_{x \to 1.5} \text{int } x = 1 = \text{int } 1.5.$$

In general, if $n - 1 < c < n$, n an integer, then

$$\lim_{x \to c} \text{int } x = n - 1 = \text{int } c.$$

Now Try Exercise 7.

Shirley Ann Jackson (1946–)

Distinguished scientist Shirley Jackson credits her interest in science to her parents and excellent mathematics and science teachers in high school. She studied physics, and in 1973, became the first African American woman to earn a Ph.D. at the Massachusetts Institute of Technology. Since then, Dr. Jackson has done research on topics relating to theoretical material sciences, has received numerous scholarships and honors, and has published more than 100 scientific articles.

Figure 2.21 is a catalog of discontinuity types. The function in (a) is continuous at $x = 0$. The function in (b) would be continuous if it had $f(0) = 1$. The function in (c) would be continuous if $f(0)$ were 1 instead of 2. The discontinuities in (b) and (c) are **removable.** Each function has a limit as $x \to 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in (d)–(f) of Figure 2.21 are more serious: $\lim_{x\to 0} f(x)$ does not exist and there is no way to improve the situation by changing f at 0. The step function in (d) has a **jump discontinuity:** The one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in (e) has an **infinite discontinuity.** The function in (f) has an **oscillating discontinuity:** It oscillates and has no limit as $x \to 0$.

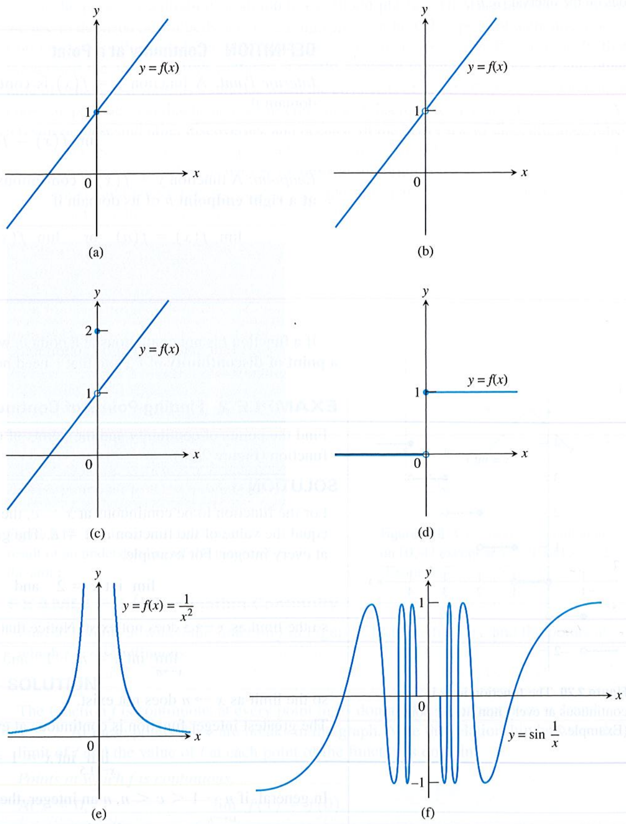


Figure 2.21 The function in part (a) is continuous at $x = 0$. The functions in parts (b)–(f) are not.

EXPLORATION 1 Removing a Discontinuity

Let $f(x) = \dfrac{x^3 - 7x - 6}{x^2 - 9}$.

1. Factor the denominator. What is the domain of f?
2. Investigate the graph of f around $x = 3$ to see that f has a removable discontinuity at $x = 3$.
3. How should f be defined at $x = 3$ to remove the discontinuity? Use ZOOM-IN and tables as necessary.
4. Show that $(x - 3)$ is a factor of the numerator of f, and remove all common factors. Now compute the limit as $x \to 3$ of the reduced form for f.
5. Show that the *extended function*

$$g(x) = \begin{cases} \dfrac{x^3 - 7x - 6}{x^2 - 9}, & x \neq 3 \\ 10/3, & x = 3 \end{cases}$$

is continuous at $x = 3$. The function g is the **continuous extension** of the original function f to include $x = 3$.

Now Try Exercise 25.

Continuous Functions

A function is **continuous on an interval** if and only if it is continuous at every point of the interval. A **continuous function** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval. For example, $y = 1/x$ is not continuous on $[-1, 1]$.

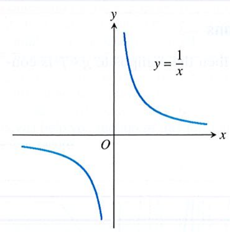


Figure 2.22 The function $y = 1/x$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$. (Example 3)

EXAMPLE 3 Identifying Continuous Functions

The reciprocal function $y = 1/x$ (Figure 2.22) is a continuous function because it is continuous at every point of its domain. However, it has a point of discontinuity at $x = 0$ because it is not defined there.

Now Try Exercise 31.

Polynomial functions f are continuous at every real number c because $\lim_{x\to c} f(x) = f(c)$. Rational functions are continuous at every point of their domains. They have points of discontinuity at the zeros of their denominators. The absolute value function $y = |x|$ is continuous at every real number. The exponential functions, logarithmic functions, trigonometric functions, and radical functions like $y = \sqrt[n]{x}$ (n a positive integer greater than 1) are continuous at every point of their domains. All of these functions are continuous functions.

Algebraic Combinations

As you may have guessed, algebraic combinations of continuous functions are continuous wherever they are defined.

THEOREM 6 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Products:* $f \cdot g$
4. *Constant multiples:* $k \cdot f$, for any number k
5. *Quotients:* f/g, provided $g(c) \neq 0$

Composites

All composites of continuous functions are continuous. This means composites like

$$y = \sin(x^2) \quad \text{and} \quad y = |\cos x|$$

are continuous at every point at which they are defined. The idea is that if $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$ (Figure 2.23). In this case, the limit as $x \to c$ is $g(f(c))$.

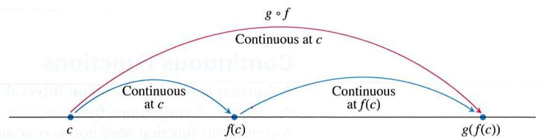
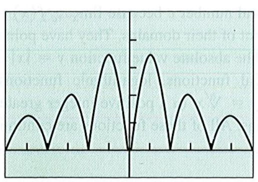


Figure 2.23 Composites of continuous functions are continuous.

THEOREM 7 Composite of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.



$[-3\pi, 3\pi]$ by $[-0.1, 0.5]$

Figure 2.24 The graph suggests that $y = |(x \sin x)/(x^2 + 2)|$ is continuous. (Example 4)

EXAMPLE 4 Using Theorem 7

Show that $y = \left|\dfrac{x \sin x}{x^2 + 2}\right|$ is continuous.

SOLUTION

The graph (Figure 2.24) of $y = |(x \sin x)/(x^2 + 2)|$ suggests that the function is continuous at every value of x. By letting

$$g(x) = |x| \quad \text{and} \quad f(x) = \frac{x \sin x}{x^2 + 2},$$

we see that y is the composite $g \circ f$.

We know that the absolute value function g is continuous. The function f is continuous by Theorem 6. Their composite is continuous by Theorem 7. ***Now Try Exercise 33.***

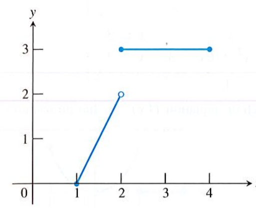


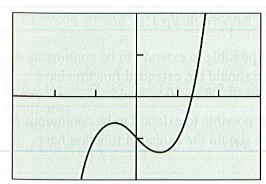
Figure 2.25 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \le x < 2 \\ 3, & 2 \le x \le 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

Grapher Failure

In connected mode, a grapher may conceal a function's discontinuities by portraying the graph as a connected curve when it is not. To see what we mean, graph $y = \text{int}(x)$ in a $[-10, 10]$ by $[-10, 10]$ window in both connected and dot modes. A knowledge of where to expect discontinuities will help you recognize this form of grapher failure.



$[-3, 3]$ by $[-2, 2]$

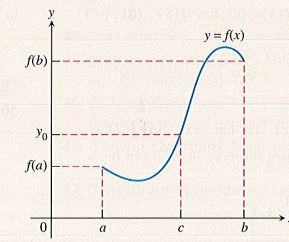
Figure 2.26 The graph of $f(x) = x^3 - x - 1$. (Example 5)

Intermediate Value Theorem for Continuous Functions

Functions that are continuous on intervals have properties that make them particularly useful in mathematics and its applications. One of these is the *intermediate value property.* A function is said to have the **intermediate value property** if it never takes on two values without taking on all the values in between.

THEOREM 8 The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



The continuity of f on the interval is essential to Theorem 8. If f is discontinuous at even one point of the interval, the theorem's conclusion may fail, as it does for the function graphed in Figure 2.25.

A Consequence for Graphing: Connectivity Theorem 8 is the reason why the graph of a function continuous on an interval cannot have any breaks. The graph will be **connected,** a single, unbroken curve, like the graph of $\sin x$. It will not have jumps like those in the graph of the greatest integer function int x, or separate branches like we see in the graph of $1/x$.

Most graphers can plot points *(dot mode).* Some can turn on pixels between plotted points to suggest an unbroken curve *(connected mode).* For functions, the connected format basically assumes that outputs *vary continuously* with inputs and do not jump from one value to another without taking on all values in between.

EXAMPLE 5 Using Theorem 8

Is any real number exactly 1 less than its cube? Compute any such value accurate to three decimal places.

SOLUTION

We answer this question by applying the Intermediate Value Theorem in the following way. Any such number must satisfy the equation $x = x^3 - 1$ or, equivalently, $x^3 - x - 1 = 0$. Hence, we are looking for a zero value of the continuous function $f(x) = x^3 - x - 1$ (Figure 2.26). The function changes sign between 1 and 2, so there must be a point c between 1 and 2 where $f(c) = 0$.

There are a variety of methods for numerically computing the value of c to be accurate to as many decimal places as your technology allows. For example, a simple application of ZOOM (box) and TRACE using a graphing calculator will quickly give the result of $c = 1.324$ accurate to three decimal places. Most calculators have a numerical zero finder that will give an immediate solution as well. ***Now Try Exercise 46.***

Can you find the *exact* value of c such that $f(c) = c^3 - c - 1 = 0$ and that you know exists by the application of the Intermediate Value Theorem? Discuss your answer with your fellow students and your teacher.

Quick Review 2.3 *(For help, go to Sections 1.2 and 2.1.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

1. Find $\lim_{x\to -1} \dfrac{3x^2 - 2x + 1}{x^3 + 4}$.

2. Let $f(x) = \text{int } x$. Find each limit.

(a) $\lim_{x\to -1^-} f(x)$ (b) $\lim_{x\to -1^+} f(x)$ (c) $\lim_{x\to -1} f(x)$ (d) $f(-1)$

3. Let $f(x) = \begin{cases} x^2 - 4x + 5, & x < 2 \\ 4 - x, & x \ge 2. \end{cases}$

Find each limit.

(a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$ (c) $\lim_{x\to 2} f(x)$ (d) $f(2)$

In Exercises 4–6, find the remaining functions in the list of functions: $f, g, f\circ g, g\circ f$.

4. $f(x) = \dfrac{2x - 1}{x + 5}$, $g(x) = \dfrac{1}{x} + 1$

5. $f(x) = x^2$, $(g\circ f)(x) = \sin x^2$, domain of $g = [0, \infty)$

6. $g(x) = \sqrt{x - 1}$, $(g\circ f)(x) = 1/x$, $x > 0$

7. Use factoring to solve $2x^2 + 9x - 5 = 0$.

8. Use graphing to solve $x^3 + 2x - 1 = 0$.

In Exercises 9 and 10, let

$$f(x) = \begin{cases} 5 - x, & x \le 3 \\ -x^2 + 6x - 8, & x > 3. \end{cases}$$

9. Solve the equation $f(x) = 4$.

10. Find a value of c for which the equation $f(x) = c$ has no solution.

Section 2.3 Exercises

In Exercises 1–10, find the points of continuity and the points of discontinuity of the function. Identify each type of discontinuity.

1. $y = \dfrac{1}{(x + 2)^2}$ **2.** $y = \dfrac{x + 1}{x^2 - 4x + 3}$

3. $y = \dfrac{1}{x^2 + 1}$ **4.** $y = |x - 1|$

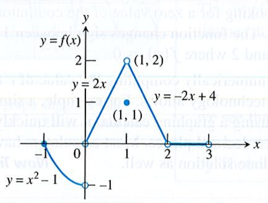
5. $y = \sqrt{2x + 3}$ **6.** $y = \sqrt[3]{2x - 1}$

7. $y = |x|/x$ **8.** $y = \cot x$

9. $y = e^{1/x}$ **10.** $y = \ln(x + 1)$

In Exercises 11–18, use the function f defined and graphed below to answer the questions.

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$



11. (a) Does $f(-1)$ exist?

(b) Does $\lim_{x\to -1^+} f(x)$ exist?

(c) Does $\lim_{x\to -1^+} f(x) = f(-1)$?

(d) Is f continuous at $x = -1$?

12. (a) Does $f(1)$ exist?

(b) Does $\lim_{x\to 1} f(x)$ exist?

(c) Does $\lim_{x\to 1} f(x) = f(1)$?

(d) Is f continuous at $x = 1$?

13. (a) Is f defined at $x = 2$? (Look at the definition of f.)

(b) Is f continuous at $x = 2$?

14. At what values of x is f continuous?

15. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?

16. What new value should be assigned to $f(1)$ to make the new function continuous at $x = 1$?

17. Writing to Learn Is it possible to extend f to be continuous at $x = 0$? If so, what value should the extended function have there? If not, why not?

18. Writing to Learn Is it possible to extend f to be continuous at $x = 3$? If so, what value should the extended function have there? If not, why not?

In Exercises 19–24, **(a)** find each point of discontinuity. **(b)** Which of the discontinuities are removable? not removable? Give reasons for your answers.

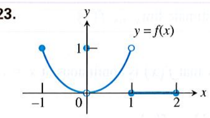
19. $f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2 \end{cases}$

20. $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ x/2, & x > 2 \end{cases}$

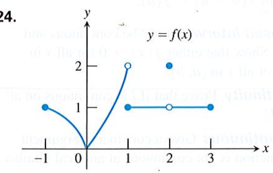
21. $f(x) = \begin{cases} \dfrac{1}{x - 1}, & x < 1 \\ x^3 - 2x + 5, & x \ge 1 \end{cases}$

22. $f(x) = \begin{cases} 1 - x^2, & x \ne -1 \\ 2, & x = -1 \end{cases}$

23.



24.



In Exercises 25–30, give a formula for the extended function that is continuous at the indicated point.

25. $f(x) = \dfrac{x^2 - 9}{x + 3}$, $x = -3$ **26.** $f(x) = \dfrac{x^3 - 1}{x^2 - 1}$, $x = 1$

27. $f(x) = \dfrac{\sin x}{x}$, $x = 0$ **28.** $f(x) = \dfrac{\sin 4x}{x}$, $x = 0$

29. $f(x) = \dfrac{x - 4}{\sqrt{x} - 2}$, $x = 4$

30. $f(x) = \dfrac{x^3 - 4x^2 - 11x + 30}{x^2 - 4}$, $x = 2$

In Exercises 31 and 32, explain why the given function is continuous.

31. $f(x) = \dfrac{1}{x - 3}$ **32.** $g(x) = \dfrac{1}{\sqrt{x - 1}}$

In Exercises 33–36, use Theorem 7 to show that the given function is continuous.

33. $f(x) = \sqrt{\left(\dfrac{x}{x + 1}\right)}$ **34.** $f(x) = \sin(x^2 + 1)$

35. $f(x) = \cos(\sqrt[3]{1 - x})$ **36.** $f(x) = \tan\left(\dfrac{x^2}{x^2 + 4}\right)$

Group Activity In Exercises 37–40, verify that the function is continuous and state its domain. Indicate which theorems you are using, and which functions you are assuming to be continuous.

37. $y = \dfrac{1}{\sqrt{x + 2}}$ **38.** $y = x^2 + \sqrt[3]{4 - x}$

39. $y = |x^2 - 4x|$ **40.** $y = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \ne 1 \\ 2, & x = 1 \end{cases}$

In Exercises 41–44, sketch a possible graph for a function f that has the stated properties.

41. $f(3)$ exists but $\lim_{x\to 3} f(x)$ does not.

42. $f(-2)$ exists, $\lim_{x\to -2^+} f(x) = f(-2)$, but $\lim_{x\to -2} f(x)$ does not exist.

43. $f(4)$ exists, $\lim_{x\to 4} f(x)$ exists, but f is not continuous at $x = 4$.

44. $f(x)$ is continuous for all x except $x = 1$, where f has a nonremovable discontinuity.

45. *Solving Equations* Is any real number exactly 1 less than its fourth power? Give any such values accurate to 3 decimal places.

46. *Solving Equations* Is any real number exactly 2 more than its cube? Give any such values accurate to 3 decimal places.

47. *Continuous Function* Find a value for a so that the function

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \ge 3 \end{cases}$$

is continuous.

48. *Continuous Function* Find a value for a so that the function

$$f(x) = \begin{cases} 2x + 3, & x \le 2 \\ ax + 1, & x > 2 \end{cases}$$

is continuous.

49. *Continuous Function* Find a value for a so that the function

$$f(x) = \begin{cases} 4 - x^2, & x < -1 \\ ax^2 - 1, & x \ge -1 \end{cases}$$

is continuous.

50. *Continuous Function* Find a value for a so that the function

$$f(x) = \begin{cases} x^2 + x + a, & x < 1 \\ x^3, & x \ge 1 \end{cases}$$

is continuous.

51. Writing to Learn Explain why the equation $e^{-x} = x$ has at least one solution.

52. *Salary Negotiation* A welder's contract promises a 3.5% salary increase each year for 4 years and Luisa has an initial salary of \$36,500.

(a) Show that Luisa's salary is given by

$$y = 36{,}500(1.035)^{\text{int } t},$$

where t is the time, measured in years, since Luisa signed the contract.

(b) Graph Luisa's salary function. At what values of t is it continuous?

53. *Airport Parking* Valuepark charge \$1.10 per hour or fraction of an hour for airport parking. The maximum charge per day is \$7.25.

(a) Write a formula that gives the charge for x hours with $0 \le x \le 24$. (*Hint:* See Exercise 52.)

(b) Graph the function in part (a). At what values of x is it continuous?

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

54. True or False A continuous function cannot have a point of discontinuity. Justify your answer.

55. True or False It is possible to extend the definition of a function f at a jump discontinuity $x = a$ so that f is continuous at $x = a$. Justify your answer.

56. Multiple Choice On which of the following intervals is $f(x) = \dfrac{1}{\sqrt{x}}$ not continuous?

(A) $(0, \infty)$ (B) $[0, \infty)$ (C) $(0, 2)$
(D) $(1, 2)$ (E) $[1, \infty)$

57. Multiple Choice Which of the following points is not a point of discontinuity of $f(x) = \sqrt{x - 1}$?

(A) $x = -1$ (B) $x = -1/2$ (C) $x = 0$
(D) $x = 1/2$ (E) $x = 1$

58. Multiple Choice Which of the following statements about the function

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -x + 3, & 1 < x < 2 \end{cases}$$

is not true?

(A) $f(1)$ does not exist.
(B) $\lim_{x \to 0^+} f(x)$ exists.
(C) $\lim_{x \to 2^-} f(x)$ exists.
(D) $\lim_{x \to 1} f(x)$ exists.
(E) $\lim_{x \to 1} f(x) \neq f(1)$

59. Multiple Choice Which of the following points of discontinuity of

$$f(x) = \frac{x(x-1)(x-2)^2(x+1)^2(x-3)^2}{x(x-1)(x-2)(x+1)^2(x-3)^3}$$

is not removable?

(A) $x = -1$ (B) $x = 0$ (C) $x = 1$
(D) $x = 2$ (E) $x = 3$

Exploration

60. Let $f(x) = \left(1 + \dfrac{1}{x}\right)^x$.

(a) Find the domain of f. (b) Draw the graph of f.

(c) **Writing to Learn** Explain why $x = -1$ and $x = 0$ are points of discontinuity of f.

(d) **Writing to Learn** Are either of the discontinuities in part (c) removable? Explain.

(e) Use graphs and tables to estimate $\lim_{x \to \infty} f(x)$.

Extending the Ideas

61. *Continuity at a Point* Show that $f(x)$ is continuous at $x = a$ if and only if

$$\lim_{h \to 0} f(a + h) = f(a).$$

62. *Continuity on Closed Intervals* Let f be continuous and never zero on $[a, b]$. Show that either $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

63. *Properties of Continuity* Prove that if f is continuous on an interval, then so is $|f|$.

64. *Everywhere Discontinuous* Give a convincing argument that the following function is not continuous at any real number.

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

2.4 Rates of Change and Tangent Lines

What you will learn about . . .

- Average Rates of Change
- Tangent to a Curve
- Slope of a Curve
- Normal to a Curve
- Speed Revisited

and why . . .

The tangent line determines the direction of a body's motion at every point along its path.

Average Rates of Change

We encounter average rates of change in such forms as average speed (in miles per hour), growth rates of populations (in percent per year), and average monthly rainfall (in inches per month). The **average rate of change** of a quantity over a period of time is the amount of change divided by the time it takes. In general, the *average rate of change* of a function over an interval is the amount of change divided by the length of the interval.

EXAMPLE 1 Finding Average Rate of Change

Find the average rate of change of $f(x) = x^3 - x$ over the interval $[1, 3]$.

SOLUTION

Since $f(1) = 0$ and $f(3) = 24$, the average rate of change over the interval $[1, 3]$ is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{24 - 0}{2} = 12.$$

Now Try Exercise 1.

Experimental biologists often want to know the rates at which populations grow under controlled laboratory conditions. Figure 2.27 shows how the number of fruit flies (*Drosophila*) grew in a controlled 50-day experiment. The graph was made by counting flies at regular intervals, plotting a point for each count, and drawing a smooth curve through the plotted points.

Secant to a Curve

A line through two points on a curve is a **secant to the curve.**

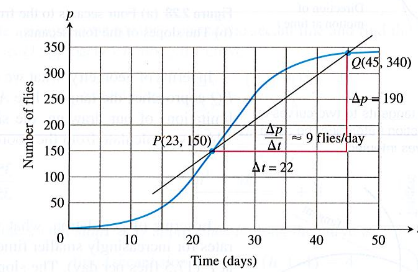


Figure 2.27 Growth of a fruit fly population in a controlled experiment. *Source: Elements of Mathematical Biology.* (Example 2)

Marjorie Lee Browne (1914–1979)

When Marjorie Browne graduated from the University of Michigan in 1949, she was one of the first two African American women to be awarded a Ph.D. in Mathematics. Browne went on to become chairperson of the mathematics department at North Carolina Central University, and succeeded in obtaining grants for retraining high school mathematics teachers.

EXAMPLE 2 Growing *Drosophila* in a Laboratory

Use the points $P(23, 150)$ and $Q(45, 340)$ in Figure 2.27 to compute the average rate of change and the slope of the secant line PQ.

SOLUTION

There were 150 flies on day 23 and 340 flies on day 45. This gives an increase of $340 - 150 = 190$ flies in $45 - 23 = 22$ days.

The average rate of change in the population p from day 23 to day 45 was

$$\textit{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day},$$

or about 9 flies per day.

continued